CONTINUOUS FAMILY OF EINSTEIN-YANG-MILLS WORMHOLES

E.E. Donets

Department of Theoretical Physics, Physics Faculty,
Moscow State University, 119899 Moscow, Russia

D.V. Gal'tsov[1]

Instituto de Física, Universidade de São Paulo,
CP 20.516, 01498-970, São Paulo, SP, Brasil

Abstract

It is shown that for some particular value of the cosmological constant depending on the gauge coupling constant a continuous one-parameter family of Einstein-Yang-Mills wormholes exists which interpolates between the instanton and the gravitating meron solutions. In contradistinction with the previously known solutions the topological charge of these wormholes is not quantized. For all of them the contribution of gravity to the action exactly cancels that of the gauge field.

[1]On leave from the Department of Theoretical Physical, Moscow State University and from the University of Campinas, Brazil.

Euclidean space-time wormholes with Yang-Mills (YM) fields [1]–[6] are obtained as solutions of the coupled Einstein-Yang-Mills (EYM) system of equations with non self-dual YM fields. In the flat euclidean space pure gauge fields can form structures known as an instanton, a meron, and nested, or dressed, merons interpolating between the first two [7]. They can be labeled by the topological charge Q that varies continuously between the value $1/2$ for the meron and 1 for the instanton. The instanton is self-dual and has a zero energy-momentum tensor, hence the gravity can be added in a self-consistent way just as a background field. The energy-momentum tensor of the meron is non-zero and this changes the situation dramatically. When gravity is coupled, the singularity at the location of a meron expands to a wormhole throat, and, consequently, the euclidean topology of the space-time transforms to that of a wormhole [1]. The value of the topological charge of the meron wormhole turns out to be zero, the charge of the meron being "swallowed" by the wormhole [3].

Unfortunately, the total action of this solution is divergent because of the slow fall-off of the meron field at infinity. To remedy this in a physically motivated way, a positive cosmological constant can be added. Then the asymptotic behavior of the field will be irrelevant since the corresponding action becomes finite simply due to the compactness of the space. Such solutions could be interpreted as describing the tunneling from the hot Friedmann-Robertson-Walker (FRW) Universe to the De Sitter one [6].

Now, the addition of the cosmological constant introduces one more new feature comparatively with the flat space case. The family of solutions with the topological charge between that of the gravitating meron ($Q = 0$) and the instanton on the FRW background ($Q = 1$) was described by Verbin and Davidson [2] and Rey [4]. These solutions could pretend to be nested meron wormholes. However, in contradistinction with the flat-space nested merons, these gravitating solutions can posses only some discrete values of the topological charge. This comes about as follows. For the isotropic homogeneous SU(2) (YM) field and the corresponding FRW metric the Einstein-Yang-Mills (EYM) system of equations reduces to a two-dimensional separable non-linear oscillator system. In order to be interpreted as a wormhole, the solution must be strictly periodic, i.e. the periods of two oscillators have to be in some rational relation. This imposes a quantization condition on the separation constant, which can be related to the topological charge. So the proposed nested meron wormholes have a quantized topological charge and consequently do not interpolate continuously between the (self-dual) instanton and the meron solutions as nested merons do in the flat space.

Here we will show, however, that some critical value of the cosmological constant depending on the gauge coupling constant exists for which both oscillators have equal periods for all values of the separation constant between those of the meron and the instanton. These wormholes can have all values of the topological charge on the interval (0, 1), and hence form indeed a continuous family of solutions interpolating between the instanton and the meron. The striking feature of these solutions is that the corresponding total action is exactly zero, the positive contribution of the YM field being canceled by the negative contribution of the gravitational field.

The critical value of the cosmological constant turns out to be the same as was found by De Alfaro, Fubini and Furlan [8] in the context of a strong gravity theory and subsequently was used in the induced gravity approach [9]–[10]. We show that following the lines of [8]–[9] a particular solution belonging to our complete set of solutions can be obtained also. However it does not have the desired wormhole properties because of inappropriate boundary conditions intrinsic to this approach.

We start with the action for the Euclidean EYM system with the cosmological constant

$$S = \frac{1}{16\pi} \int \sqrt{g}\, (m_{p\ell}^2 (2\lambda - R) + F_{\mu\nu}^a\, F^{a\mu\nu})\, d^4x \quad , \tag{1}$$

assuming the metric to have the form

$$ds^2 = N^2\, d\tau^2 + a^2 (d\chi^2 + \sin^2\chi\, (d\theta^2 + \sin^2\theta d\varphi^2)) \quad . \tag{2}$$

For the SU(2) YM connection (generalization to higher groups is straightforward along the lines of [5]–[6]) symmetric under the space-time isometry group SO(4) one can use either a temporary gauge [1]–[2] or a Witten ansatz, the relationship between the two having been clarified in [11]. The field parametrized by a single real-valued function s of the euclidean time τ in the temporary gauge reads

$$\begin{aligned} A_0 &= 0 \quad , \qquad eA_j = \frac{i}{2}\, (\sigma(\tau) + 1)\, U \partial_j U^{-1} \quad ; \\ U &= \exp\left[i\chi\, (\sin\theta(\sigma_x \cos\varphi + \sigma_y \sin\varphi) + \sigma_z \cos\theta)\right] \quad . \end{aligned} \tag{3}$$

Integrating out the angle variables in (1) and eliminating the total derivative one gets $S = S_{gr} + S_{YM}$,

$$S_{YM} = \frac{3\pi}{4e^2} \int \left(\frac{\dot{\sigma}^2 a}{N} + (\sigma^2 - 1)^2\, \frac{N}{a} \right) d\tau \quad , \tag{4.a}$$

$$S_{gr} = \frac{3\pi m_{p\ell}^2}{4} \int \left(\frac{-\dot{a}^2 a}{N} - aN + \frac{\lambda}{3}\, Na^3 \right) d\tau \quad , \tag{4.b}$$

where dot denotes the derivative with respect to τ. The corresponding equations of motion for the YM variable $\sigma(\tau)$ and the radius $a(\tau)$ of the Universe read

$$\frac{d}{d\tau} \left(\frac{\dot{\sigma} a}{N} \right) - 2\sigma\, (\sigma^2 - 1)\, \frac{N}{a} = 0 \quad , \tag{5.a}$$

$$\frac{d}{d\tau} \left(\frac{\dot{a} a}{N} \right) - N + \frac{2}{3}\, \lambda N a^2 = 0 \quad . \tag{5.b}$$

Variation with respect to the lapse function gives the constraint

$$m_{p\ell}^2 \left(\frac{a\dot{a}^2}{N^2} - a + \frac{2}{3}\,\lambda a^3 \right) + \frac{1}{e^2} \left(-\frac{\dot{\sigma}^2 a}{N^2} + \frac{(\sigma^2-1)^2}{a} \right) = 0 \tag{6}$$

which is just the $\tau\tau$ - component of the Einstein equations. The non-linear two-oscillator system (5) separates in the conformal gauge $N = a$:

$$\ddot{\sigma} - 2\sigma\,(\sigma^2 - 1) = 0 \tag{7.a}$$

$$\ddot{a} - a + \frac{2}{3}\,\lambda a^3 = 0 \tag{7.b}$$

while the constraint equation (6) then just equates the first integrals of (7)

$$\dot{\sigma}^2 + U_\sigma = -C \quad , \qquad U_\sigma = -\,(\sigma^2-1)^2 \quad , \tag{8.a}$$

$$\dot{a}^2 + U_a = -\frac{C}{m_{p\ell}^2 e^2} \quad , \qquad U_a = \frac{\lambda a^4}{3} - a^2 \quad . \tag{8.b}$$

Here C is the separation constant, restricted to the interval $[0,1]$ for the wormhole solutions [2].

The quatric potentials U_σ and U_a have a similar shape and possess four zeros

$$U_\sigma\,(\pm\sigma_\pm) = 0 \quad , \qquad \sigma_\pm = (1 \pm C^{1/2})^{1/2} \quad , \tag{9.a}$$

$$U_a\,(\pm a_\pm) = 0 \quad , \qquad a_\pm = \left[\frac{3}{2\lambda} \left(1 \pm \left(1 - \frac{4\lambda C}{3 m_{p\ell}^2 e^2} \right)^{1/2} \right) \right]^{1/2} , \tag{9.b}$$

satisfying the following relations

$$\sigma_+^2 + \sigma_-^2 = 2 \quad , \qquad \sigma_+\sigma_- = (1-C)^{1/2} \quad , \tag{10.a}$$

$$a_+^2 + a_-^2 = \frac{3}{\lambda} \quad , \qquad a_+^2 a_-^2 = \frac{3C}{\lambda m_{p\ell}^2 e^2} \quad . \tag{10.b}$$

The actual motion for wormhole solutions is restricted to the intervals $-\sigma_- < \sigma < \sigma_-$, $a_- < a < a_+$, and the boundary conditions have to be of the following form

$$\sigma(0) = -\sigma_- \quad , \quad \dot\sigma(0) = 0 \;\; ; \quad \sigma(\tau_f) = \sigma_- \quad , \quad \dot\sigma(\tau_f) = 0 \quad ; \tag{11.a}$$

$$a(0) = a_- \quad , \quad \dot a(0) = 0 \;\; ; \quad a(\tau_f) = a_+ \quad , \quad \dot a(\tau_f) = 0 \tag{11.b}$$

for some moment τ_f of the conformal time τ. This is, of course, an overdetermined system, which can not be generally satisfied, thus leading to the above quantization condition. Indeed, general solutions of the equations (7), (8) satisfying the initial conditions (11a) at the left turning points can be expressed in terms of the elliptic functions as follows

$$\sigma(\tau) = \sigma_- \mathrm{sn}\left[\sigma_+ \left(\tau - \frac{T_\sigma}{2}\right), k_\sigma^2\right] \quad , \qquad k_\sigma = \frac{\sigma_-}{\sigma_+} \quad ; \tag{12.a}$$

$$a^2(\tau) = a_+^2 + (a_-^2 - a_+^2)\,\mathrm{sn}^2\left[\frac{a_+}{\sqrt{a_+^2 + a_-^2}}\,(\tau - T_a)\,, k_a^2\right] \quad ,$$

$$k_a = \frac{\sqrt{a_+^2 - a_-^2}}{a_+} \quad . \tag{12.b}$$

The conditions (11b) at the right turning points, however, can only be satisfied if $\tau_f = n_\sigma T_\sigma$ and $\tau_f = n_a T_a$ where T_σ and T_a are proper half-periods for the corresponding variables, and $n_\sigma, n_a \in \mathbb{N}$. The periods can be expressed in terms of the complete elliptic integrals of the first kind

$$T_\sigma = \frac{2}{\sigma_+} K\left(k_\sigma\right) \quad , \tag{13.a}$$

$$T_a = \frac{\sqrt{a_+^2 + a_-^2}}{a_+} K\left(k_a\right) \quad , \tag{13.b}$$

and we thus get the quantization condition of Verbin and Davidson [2] for the separation constant C

$$n_\sigma \, T_\sigma = n_a \, T_a \quad . \tag{14}$$

Let us show that the Eq.(14) can be thought of as a quantization condition for the topological charge

$$Q = \frac{e^2}{64\pi^2} \int_0^{T_\sigma} d\tau \int_{S^3} e^{\mu\nu\lambda\tau} \, F^a_{\mu\nu} \, F^a_{\lambda\tau} \, d^3x = P(T_\sigma) - P(0) \quad , \tag{15}$$

where

$$P(\tau) = \frac{e^2}{32\pi^2} \int_{S^3} e^{o\mu\nu\lambda} \, A^a_\mu \left(\partial_\nu A^a_\lambda + \frac{e}{3} \, e^{abc} \, A^b_\nu \, A^c_\lambda \right) d^3x \tag{16}$$

is the Chern-Simons number. Substituting the Eq.(3) we get

$$P(\tau) = \frac{1}{4} \left(3\sigma + 2 - \sigma^3\right) \quad , \tag{17}$$

and, hence,

$$Q = \frac{1}{2} \, \sigma_- \left(3 - \sigma_-^2\right) \quad . \tag{18}$$

Because of the quantization of C, this quantity will have also discrete values.

Now, if we take into account the following transformation formula for the complete elliptic integrals of the first kind

$$K(2\sqrt{z}/(1+z)) = (1+z)\,K\,(z) \tag{19}$$

we can easily realise that the necessary relation between the moduli k_σ and k_a of the elliptic functions, entering the formulas (13) can be satisfied indeed if only the turning points are related by

$$m_{p\ell}\,e\,a_\pm = \sigma_+ \pm \sigma_- \quad . \tag{20}$$

This fix the value of the cosmological constant as follows

$$\lambda = \lambda_{cr} = \frac{3}{4}\,m_{p\ell}^2\,e^2 \quad . \tag{21}$$

For this particular value of the cosmological constant the relation

$$k_a = \frac{2\sqrt{k_\sigma}}{1+k_\sigma} \tag{22}$$

holds for all $C \in [0,1]$, and the half-periods exactly coincide $T_\sigma = T_a = T$. It is interesting to note that for such critical λ the radius a becomes constant for $C = 1$, i.e. in the meron limit. Hence in our case the meron wormhole metric transforms into that of the Euclidean static Einstein Universe. For C non equal to $1, 0$ the solution (12) preserves its meaning of a wormhole.

Let us calculate the value of the action integral between $\tau = 0$ and $\tau = T$. Changing the integration variable in the Eq.(4a) to σ one can easily obtain the desired quantity in terms of the complete elliptic integrals of the first and the second kind

$$
\begin{aligned}
S_{YM} &= \frac{3\pi}{4e^2}\int_0^T \left(\dot{\sigma}^2 - U_\sigma\right) d\tau \\
&= \frac{2\pi}{e^2}\left(\sigma_+ E(k_\sigma) - \left(C + \frac{\sqrt{C}}{4}\right)\frac{1}{\sigma_+} K(k_\sigma)\right) \quad . \qquad (23)
\end{aligned}
$$

Similarly, the gravitational part of the action is calculated passing to the integration variable a

$$
\begin{aligned}
S_{g\tau} &= \frac{3\pi m_{p\ell}^2}{4}\int_0^T \left(-\dot{a}^2 + U_a\right) d\tau \\
&= \sqrt{\frac{3}{\lambda}}\,\frac{\pi m_{p\ell}^2}{2}\left(-a_+ \, E\left(k_a\right) + \frac{1}{2m_{p\ell}^2 e^2 a_+}\, C\, K\left(k_a\right)\right) \quad . \qquad (24)
\end{aligned}
$$

Now, for the critical cosmological constant (21) the relation (22) between the arguments of elliptic integrals holds. To relate the values (23) and (24) one has to use the transformation formula (19) together with the corresponding formula for the elliptic integrals of the second kind ($0 \le z \le 1$):

$$
(1+z)\, E\left(\frac{2\sqrt{z}}{1+z}\right) = 2\, E(z) - (1 - z^2)\, K(z) \quad . \qquad (25)
$$

This gives precisely

$$
S_{YM} + S_{gr} = 0 \quad . \qquad (26)
$$

To get more physical insight into the nature of the critical λ given by the Eq.(21) we turn to the discussion of the conformal properties of the solutions in question. Since the YM equations are conformally invariant, and we are considering conformally flat (in fact conformal to the static Einstein Universe) metrics, one could start with the YM theory in the flat Euclidean space-time. Then the standard Corrigan-Fairly-'t Hooft-Wilczek ansatz (in cartesian coordinates t, x, y, z)

$$A_\mu = \frac{i}{e}\,\sigma_{\mu\nu}\,\partial^\nu \ln h \quad , \tag{27}$$

where $\sigma_{oi} = \dfrac{\sigma_i}{2}, \sigma_{ij} = \dfrac{1}{2}\epsilon_{ijk}\,\sigma^k$, provides a solution of the sourceless YM equation if the following equation for the scalar function h holds

$$\Box h + \kappa\, h^3 = 0 \quad , \tag{28}$$

where κ is some (still arbitrary) constant, and $\Box$ is a flat-space four-dimensional Laplacian. Now, as De Alfaro, Fubini and Furlan had found [8] in the context of the strong gravity, the theory may be put into a larger curved space context by considering confomally flat metrics $g_{\mu\nu} = h^2\,\delta_{\mu\nu}$ with the same function entering as a conformal factor. The corresponding scalar curvature will be

$$R = -\frac{6\Box h}{h^3} \quad . \tag{29}$$

Then taking into account the tracelessness of the YM energy-momentum tensor, one gets from the Einstein equations with the cosmological constant

λ the following equation for h:

$$\Box h + \frac{2}{3}\,\lambda h^3 = 0 \tag{30}$$

This is just the equation (28) with a particular value of the constant $\kappa = 2\lambda/3$. Now, from the gravitational Hamiltonian constraint equation we get exactly our relation between the cosmological constant and the gauge coupling constant (21).

To relate this to our previous considerations, let h be a function of the variable $\tau = (t^2 + x^2 + y^2 + z^2)^{1/2}$ and make the following coordinate transformation

$$\begin{aligned} t &= \cos\chi \exp(\tau) \quad , \\ x &= \sin\chi \cos\varphi \sin\theta \exp(\tau) \quad , \\ y &= \sin\chi \sin\varphi \sin\theta \exp(\tau) \quad , \\ z &= \sin\chi \cos\theta \exp(\tau) \quad . \end{aligned} \tag{31}$$

This will reproduce our previous ansatze for the metric and the YM connection (2), (3) with

$$\sigma = -1 - \dot{h}/h \quad , \qquad a = h \exp(\tau) \tag{32}$$

where a dot denotes the derivative with respect to τ as before. From here it follows immediately that

$$\sigma = -\frac{\dot{a}}{a} \tag{33}$$

Since both the conformal factor and the YM connection were now described in terms of the only function h, it is clear that both variables will have the same periods for periodic solutions. Hence we have reproduced the desired property imposed by the wormhole boundary conditions (11) from an alternative point of view. However, the equation (33) selects from the whole set of solutions of the system (7), (8) a particular one, which can not be interpreted itself as a wormhole, because the initial conditions according to the Eq (29) will be $\sigma(0) = -\dot{a}(0)/a(0)$ (i.e. when thy conformal factor starts tunneling, the YM field starts exactly in the middle under the potential barrier). Yet this particular solution seems to be interesting by itself. Amazingly enough it can be described entirely in terms of elementary functions. Let us use comoving coordinates in (2) putting in the Eqs(5) $N = 1$. Making a substitution $f = a^2$ we will get a linear equation

$$\frac{d^2 f}{d\tau_c^2} - 2 + (em_{p\ell})^2 f = 0 \tag{34}$$

which is solved with initial conditions $f(0) = a_-^2, \dot{f}(0) = 0$ as follows

$$f = \frac{a_+^2 + a_-^2}{2} - \frac{a_+^2 - a_-^2}{2} \cos(em_{p\ell}\tau_c) \quad . \tag{35}$$

The corresponding YM function can then be obtained from Eq.(29):

$$\sigma = -\frac{1}{2\sqrt{f}} \frac{df}{d\tau_c} \quad . \tag{36}$$

To summarize: we have shown, that a continuous family of EYM wormholes with the gauge group SU(2) exists for some particular value of the cosmological constant depending on the gauge coupling constant. The absolute value of the total action in this case is minimal and moreover, exactly equal to zero. The meron wormhole then transforms into the static Einstein Universe, while nested merons correspond to wormholes possessing all values of a topological charge between 0 and 1. Cosmological implications of the results obtained will be discussed in a separate publication.

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